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ON SYNTHESIS OF STABILITY OF SYSTEMS BY THE METHOD OF NONLINEAR PROGRAMING

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A method of successive computation of the parameters governing the equation for stabilization of linear systems based on the ideas of nonlinear programing and reducing to minimization of the original functional is described. We do not succeed in presenting a rigorous mathematical foundation.

1. Let the perturbed motion of a stationary linear control system be described by the set of differential equations

$$dX/dt = AX + BU \tag{1.1}$$

Here X is the column vector of the fundamental variables; A is a square ($n \times n$) matrix, B is the column vector of the control efficiency coefficients, and U is a scalar of the controlling effect of the regulator.

It is assumed that the system (1.1) satisfies the controllability conditions. The matrices A, B are not degenerate, and the matrix $\varphi = \| B, AB, A^2B, \ldots, A^{n-1}B \|$ is of rank n and consists of n linearly independent vectors. It is required to seek the control law

$$U = CX \tag{1.2}$$

assuring asymptotic stability of the unperturbed motion $X = 0$. It is assumed that the matrix C has the form of a row vector and yields a square ($n \times n$) matrix in the product BC . Substituting (1.2) into the system (1.1) we obtain

$$dX/dt = (A + BC)\,X \tag{1.3}$$

To solve the stabilization problem, let us utilize the second Liapunov method. We construct the quadratic form V; as

$$V = X^{*}KX \tag{1.4}$$

Here K is a symmetric square ($n \times n$) matrix. Let W be the total derivative

$$W = X^{*\cdot}KX + X^{*}KX^{\cdot} \tag{1.5}$$

After substituting (1.3) into (1.5) and multiplying by -1 we obtain

$$-W = X^{*}(T^{*} + T)X, \qquad (T = -K(A + BC)) \tag{1.6}$$

The matrices A, B are known in the stabilization problem. Let us assign the matrix K. Let us introduce the penalty function of the form

$$M = \sum_{k=1}^{2n} \exp(-M_{kk}\Phi) \tag{1.7}$$

Here M_{kk} are the principal diagonals of the minors of the discriminants of the quadratic forms ($-W$) and V; a factor Φ exceeding $+1$ is introduced to accelerate the computations, and n is the order of the system (1.1). The penalty function M has an extremum within the domain of asymptotic stability. The dependence of the penalty function on the principal diagonals of the minors will evidently be as shown in Fig. 1, where L is the domain of asymptotic stability. A discrete or continuous algorithm can be utilized to minimize the penalty function M of the row vector C. The discrete algorithm is described by the recursion relationship

$$C_{n+1} = C_n - \operatorname{grad} M_c \cdot h \quad (h \text{ is the spacing}) \tag{1.8}$$

Here grad M_c is the gradient of the penalty function with respect to the vector C. The continuous algorithm is described by the system of differential descent equations in the domain of asymptotic stability

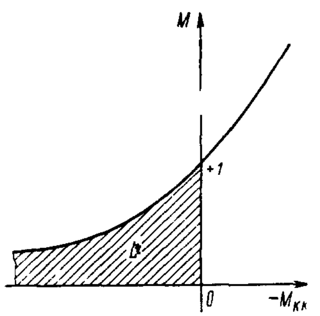

Fig. 1

$$dC/d\tau = -\partial M/\partial C \quad (\tau \text{ is auxiliary time}) \tag{1.9}$$

The initial conditions can be given as random numbers. The boundary of the asymptotic stability domain L is equivalent to the fact that each member of (1.7) for the penalty function equals $+1$ since $\exp 0 = +1$. The Sylvester inequalities can be satisfied by fixing the magnitude of each member of M.

To enter the domain L it is sufficient to seek such values of the vector C by means of the algorithms (1.8) or (1.9), for which each member of M would be less than $+1$. All other stability inequalities can be utilized in addition to Sylvester inequalities. Utilizing the Hurwitz inequalities, the penalty function has the form

$$M = \sum_{k=1}^{n} \exp(-H_{kk}\Phi) \tag{1.10}$$

Here H_{kk} are the principal diagonals of the minors of the Hurwitz matrix. Accession to the domain L by means of the continuous and discrete algorithms is guaranteed only under the condition that it exists.

The algorithms of descent into the domain L, (1.8) and (1.9), can be used to solve optimal stabilization problems. They permit successive seeking for such values of the control law parameters in the system of quadratic equations ([1], p. 495) for which stability and optimality of the control are guaranteed. To do this it is necessary to supplement the penalty function M by the sum of squares of residuals of the system of quadratic equations

$$S = M + \rho = M + \varepsilon_1^2 + \ldots + \varepsilon_n^2 \qquad (1.11)$$

and then to minimize S by means of the algorithms (1. 8) or (1. 9). The minimization process is continued even after entrance into the domain L until sufficient accuracy of the optimization is obtained.

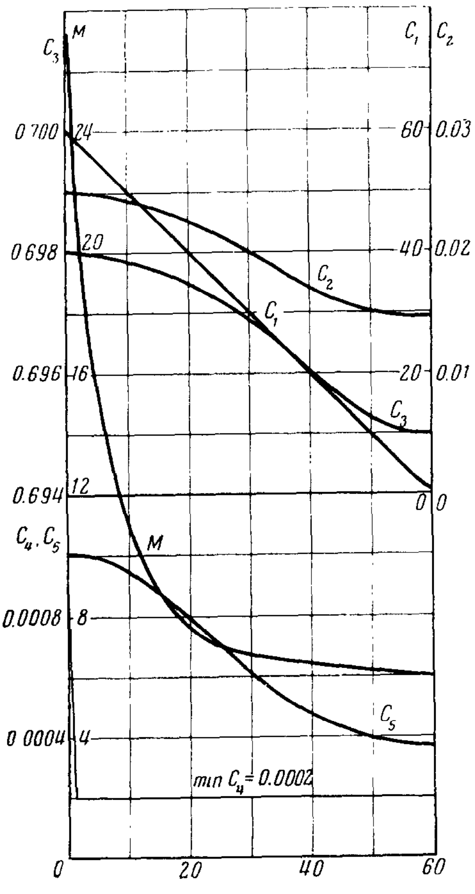

Fig. 2

2. To illustrate the synthesis of stable systems by the method of nonlinear programing, let us consider a linear sixth order system with five control parameters.

The characteristic polynomial of the automatic system is

$$A_0x^6 + A_1x^5 + A_2x^4 + A_3x^3 + A_4x^2 + A_5x + A_6 = 0 \qquad (2.1)$$

The coefficients of the characteristic polynomial depend on the parameters of the automatic system

$$A_0 = k_6C_3k_2k_3 \qquad (2.2)$$

$$A_1 = (k_1k_3 + k_5k_2)\, C_1C_4C_5k_6 + (k_3 + k_2k_4)\, k_6C_3 + (C_3 + k_6C_2)\, k_2k_3$$

$$A_2 = (k_1k_4 + k_5)\, C_1C_4C_5k_6 + k_6C_3\,(k_2 + k_4) + (k_3 + k_2k_4)\,(C_3 + k_6C_2) + (C_2 + k_6)\, k_2k_3$$

$$A_3 = C_1\,((C_4 + C_5)\,(k_1k_3 + k_5k_2) + k_1C_4C_5k_6 + C_5\,(k_6 + C_4)\,(k_1k_4 + k_5)) + (k_2 + k_4)\,(C_3 + k_6C_2) + k_6C_3 + (k_3 + k_2k_4)\,(C_2 + k_6) + k_2k_3$$

$$A_4 = k_3 + k_2k_4 + (k_4 + k_2)\,(C_2 + k_6) + C_3 + k_6C_2 + C_1.(k_1C_5\,(C_4 + k_6) + (C_5 + C_4)\,(k_1k_4 + k_5) + k_1k_3 + k_5k_2)$$

$$A_5 = C_1\,(k_1k_4 + k_5 + k_1\,(C_4 + C_5)) + C_2 + k_6 + k_4 + k_2, \qquad A_6 = C_1k_1 + 1$$

From the conditions of the problem the coefficients k_i had the following values:

$$K_1 = 1.32,\ K_2 = 0.00157,\ K_3 = 0.000197,\ K_4 = 0.00019,\ K_5 = 0.93,\ K_6 = 0.0064 \qquad (2.3)$$

The control parameters of the automatic system C_i are to be determined. In the zero approximation it was assumed that

$$C_{10} = 60{,}0, \quad C_{20} = 0.02, \quad C_{30} = 0.7, \quad C_{40} = C_{50} = 0.001 \qquad (2.4)$$

For these values of the control parameters, the characteristic polynomial of the automatic system (2. 4) is not a Hurwitz polynomial; hence, the penalty function is $M_0 = 27.948$.

To descend into the stability domain, the standard ALGOL program for searching for the extremum of a function of five arguments by the gradient method was used. The penalty function (1.10), depending on the Hurwitz inequalities, was minimized by the discrete algorithm in the program.

The lower bound $C_{i\,\min} = 0.0002$, was imposed on the control parameters since they should not be negative. The factor Φ to accelerate the computations was taken equal to $\Phi = 10^{+15}$ for values of $H_{kk} < +1$.

The descent trajectory of the five control parameters of the automatic system and the stability domain were computed on the BESM-4 computer. The results of the computation are shown in Fig. 2. The values obtained for the control parameters are

$$C_1 = 0.085614, \quad C_2 = 0.005035, \quad C_3 = 0.697956, \quad C_4 = 0.0002, \quad C_5 = 0.000367 \qquad (2.5)$$

The penalty function is $\mathbf{M} = 5.99939$. For these values of the control parameters the characteristic polynomial is a Hurwitz polynomial; its roots are

$$\lambda_6 = -0.0064, \qquad \lambda_5 = -0.001569 \qquad (2.6)$$

$$\lambda_{4,3} = -0.037662 \pm i\, 0.790725, \qquad \lambda_{2,1} = -0.000085 \pm i\, 0.014039$$

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ON A THEOREM OF EXISTENCE OF A PERIODIC SOLUTION TO THE LIÉNARD EQUATION

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The criterion of existence of a periodic solution of the Liénard equation

$$x^{\cdot\cdot} + f(x)\, x^{\cdot} + g(x) = 0$$

is established. Definite constraints are imposed on the functions $f(x)$ and $g(x)$, but only for a certain, sufficiently wide range of the values of x, containing the coordinate origin.

Let us replace the given equation with an equivalent system given by

$$dx \,/\, dt = y, \qquad dy \,/\, dt = -yf(x) - g(x) \qquad (1)$$

and introduce the notation

$$F(x) = \int_0^x f(x)\, dx, \qquad G(x) = \int_0^x g(x)\, dx, \qquad Q(x) = 2G(x) - {}^1/_4\, \lambda^2 x^2 + \lambda \int_0^x F(x)\, dx$$

$$p(x) = 2F(x) - \lambda x, \qquad r(x) = 2G(x) + F^2(x) - \lambda x F(x) + \lambda \int_0^x F(x)\, dx$$

where λ is any positive real number for which the conditions of the theorem hold.